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COMMENT

A note on the contraction of Lie algebras

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Abstract. It is demonstrated that the contraction of Lie algebras can be viewed as the procedure that underlies limiting distributions in probability theory. The consequences of such an interpretation are discussed.

The contraction of Lie algebras, i.e. the method of obtaining one Lie algebra from another Lie algebra (usually non-isomorphic) by means of a limiting procedure, was originally introduced by Inonu and Wigner (1953) and later developed by Saletan (1965) and it is discussed elaborately by many authors (Venkatesan 1967, Gilmore 1974, Barut and Raczka 1980). Arecchi *et al* (1972) employed this method to obtain the harmonic oscillator coherent states from the so-called atomic coherent states. In this comment we shall explicitly show, using two examples, that the procedure involved in the contraction of Lie algebras is closely related to the well known method of obtaining one probability distribution from another probability distribution involving a limiting procedure.

The coherent states of the harmonic oscillator algebra (Klauder and Sudarshan 1968) (also known as Heisenberg-Weyl algebra) and angular momentum algebra (also known as SU(2) algebra) (Radcliffe 1971, Arecchi *et al* 1972) have been defined by different people in different contexts. The coherent state representation of Lie groups is also well studied (Hioe 1974, Onofri 1975). Coherent states that arise from the Heisenberg-Weyl algebra are known to be the eigenstates of the destruction operator a:

$$a|z\rangle = z|z\rangle \tag{1}$$

where z is a complex number and its Fock space representation is given by

$$|z\rangle = \exp(-|z|^2/2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle. \tag{2}$$

These states have many interesting properties and applications, especially in quantum optics (Klauder and Sudarshan 1968). Also

$$f_n(z) = |\langle n|z\rangle|^2 = \exp(-|z|^2)\frac{(|z|^2)^n}{n!} \tag{3}$$

gives the probability that there are n photons in the coherent state $|z\rangle$. Due to equation (3), $|z\rangle$ is known as the Poissonian superposition of number states $|n\rangle$.

Coherent states of angular momentum are defined as (Radcliffe 1971, Arecchi *et al* 1972, Hioe 1974)

$$|\mu\rangle = \frac{1}{(1+|\mu|^2)^j} \sum_{p=0}^{2j} \binom{2j}{p}^{1/2} \mu^p |p\rangle \tag{4}$$

where μ is a complex number and $|p\rangle$ are the projections of a single angular momentum j.

Owing to apparent similarities in the treatment of coherent states of the harmonic oscillator and angular momentum, Arecchi *et al* (1972) showed, by using contraction of Lie algebras, that the angular momentum coherent states go over to the harmonic oscillator coherent states. Not much has been known about the meaning of this contraction procedure. We propose that this limiting procedure could be *understood* in the language of probability theory as illustrated below. Now

$$\pi_p(\mu)=|\langle p|\mu\rangle|^2=\binom{2j}{p}(|\mu|^2)^p(1+|\mu|^2)^{-2j} \tag{5}$$

gives the probability that a system described by the coherent state $|\mu\rangle$ is in the projected state $|p\rangle$ which is a binomial distribution.

Taking $|\mu|^2=|z|^2/2j$ equation (5) can be written as

$$\pi_p(\mu)=\frac{(2j)(2j-1)\dots(2j-p+1)}{[1+(|z|^2)/2j]^{2j}}\frac{(|z|^2)^p}{p!(2j)^p}. \tag{6}$$

If in equation (6) we keep z fixed and let j tend to infinity then $\pi_p(\mu)\to f_p(z)$ of equation (3). This is the so-called Holstein–Primakoff (1940) limit used by Arecchi *et al* (1972). Thus the *contraction of Lie algebras* used by Arecchi *et al* (1972) entails a *contraction of probability distribution.* Here we mean that the 'contraction of probabilities' is the limit involved in one distribution going over to another distribution.

In the case of SU(1, 1) the commutation relations are specified by

$$\begin{aligned}&[J_3, J_\pm]=\pm J_\pm\\&[J_-, J_+]=2J_3\end{aligned} \tag{7}$$

where $J_\pm$ are the ladder operators. Using Perelemov's definition (Barut and Girardello 1971, Perelemov 1977) the coherent states are defined as

$$|n, z, \mu\rangle=\sum_{k=0}^{\infty}\binom{n+k-1}{k-1}^{1/2}z^n\mu^k|k\rangle \tag{8}$$

where z and μ are complex and related by $|z|^2+|\mu|^2=1$. The associated probability distribution is given by

$$P_k(\mu)=\binom{n+k-1}{k-1}(|z|^2)^n(|\mu|^2)^k \tag{9}$$

which is negative binomial.

If in equation (9) we let $|\mu|^2$ tend to zero, n tend to infinity and $|\mu|^2n$ tend to λ then $P_k(\mu)\to f_k(\lambda)$ of equation (3). This is the same limiting procedure employed by Barut and Girardello (1971). So the contraction of SU(1, 1) to Heisenberg–Weyl algebra is the same as the contraction of the negative binomial distribution associated with SU(1, 1) to the Poisson distribution associated with Heisenberg–Weyl algebra.

This relationship between Lie algebraic contraction and 'contraction of probabilities' could be extended to the general theory of contraction of Lie algebras. Probability distributions could be associated with arbitrary Lie algebras via defining coherent states as shown by Perelemov (1977). This makes the study of Lie algebras interesting in the same way as special functions are associated with them (Miller 1968). The connection between 'contraction of probabilities' and the study of Cayley–Klein geometries (Sanjuan 1984) via the group contraction procedure will be published elsewhere.

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